Higher order infinity regularizations in gravity-modified quantum electrodynamics

1973 J. Phys. A: Math. Nucl. Gen. 6 999

(http://iopscience.iop.org/0301-0015/6/7/020)

J. Phys. A: Math., Nucl. Gen., Vol. 6, July 1973. Printed in Great Britain.

Higher order infinity regularizations in gravity-modified quantum electrodynamics†

Suraiya Parveen
Department of Physics, University of Karachi, Karachi, Pakistan

Received 29 January 1973

Abstract. The fermion self-mass is computed to order e^4 in gravity-modified quantum electrodynamics. It is found that regularization does proceed as one would expect from the results of second order computations given by Abdus-Salam, Isham and Strathdee. The inverse of the gravitational constant which appears as an inbuilt cut-off regularizes the infinite terms $(e^2 \ln \infty)^2$ and $e^4 \ln \infty$, respectively, to $(e^2 \ln 1/\kappa m)^2$ and $e^4 \ln 1/\kappa m$.

1. Introduction

The modification of quantum electrodynamics proposed by Abdus-Salam *et al* (1971) yields, as they have shown, a finite result for the self-mass of order e^2. In fact, the consideration of gravity in conventional quantum electrodynamics has given rise to a self-consistent relativistic quantum field theory; where to our present knowledge every physical process involving electromagnetic interaction can be quantitatively studied in principle to any given order of approximation.

In the past theories were formulated based on the principle of the special theory of relativity. The formulation of more general and covariant theories based on the principle of general relativity was avoided only because of the lack of mathematical techniques available at that time. The general covariance demanded the action integral $\int \mathscr{L}(x)\, d^4x$ to be invariant under general coordinate transformation. It was only possible when the lagrangian carried a factor $(\det L^{\mu a})^{-1}$; $L^{\mu a}$ is the Vierbein gravity field. This factor made every lagrangian non-polynomial. Because of its complexity and some mathematical difficulties we neglected this approach without realizing that this could distort space–time and lead to non-physical results.

Recent development of new mathematical techniques for handling the non-polynomial lagrangians (Okubo 1954, Efimov 1963, Fradkin 1963, Volkov 1968, Abdus-Salam *et al* 1970) has given hope for the construction of a general covariant theory. The typical characteristic of the non-polynomial theories is found to be the logarithmic dependence of the Green function on its coupling constant. Thus in the non-polynomial theory of electromagnetic interactions we get a term proportional to $(\kappa^2 p^2)^n \ln(\kappa^2 p^2)$; the gravitational coupling constant κ is related to the newtonian gravitational constant G_N by $\kappa^2 = 8\pi G_N = 10^{-22} m_e^{-1}$. Neglecting gravity, by setting $G_N = 0$ we notice that it was this logarithmic term which behaved as $\ln \infty$ in conventional quantum electrodynamics. To remove these singularities we adopted a purely mathematical regularization; we cut-off the divergent integrals by introducing a highly massive field without any physical interpretation.

† This work was submitted to London University in partial fulfilment of the requirements for a PhD degree.

In this new model we find that the inverse of the gravitational constant ($\kappa^{-1} = 10^{19}$ BeV) appears as an inbuilt cut-off. Here gravity plays the role of a regulator at high frequencies. Physically it can be interpreted as if the electron has an intrinsic radius equal to its Schwartzschild radius $r = 2m_e^2 G_N$, measured in units of m_e^{-1}. This provides a natural cut-off for the wavelength of any virtual photon which the electron can emit.

In the next section we will give a brief outline of the techniques involved; for details we refer to the paper of Abdus-Salam *et al* 1(971). We then proceed to the fourth order computation of the fermion self-mass. In general one does not need the mass correction of order e^4 for any physical purpose, but we thought it worthwhile to have a further check on the consistency of the theory by going up to higher orders.

2. Gravity-modified electromagnetic interaction

The interaction part of the gravity-modified lagrangian is given by

$$\mathscr{L}_{\text{int}}(x) = e_0 \frac{L^{\mu a}(x)}{\det L} \bar{\psi}\gamma_a \psi A_\mu . \tag{1}$$

In the limit of asymptotically flat space–time the field $L^{\mu a}$ can be expressed in terms of the physical graviton field $\phi^{\mu a}$

$$L^{\mu a}(x) = \eta^{\mu a} + \tfrac{1}{2}\kappa\phi^{\mu a}(x) \tag{2}$$

where $\eta^{\mu a} = \text{diag}(1, -1, -1, -1)$. Here $(\det L)^{-1}$ is the main source of non-polynomiality. By setting $\kappa = 0$, the above lagrangian reduces to the conventional Dirac lagrangian.

Formally $\mathscr{L}_{\text{int}}$ can be expressed in the form of a power series,

$$\mathscr{L}_{\text{int}}(x) = e_0 \bar{\psi}\gamma_a \psi A_\mu \left(\sum_{n=0}^{\infty} c(n)(\phi^{\mu a})^n \kappa^n \right). \tag{3}$$

According to this expression at each point of space–time there will be emission or absorption from an electron line of a photon and an infinite number of gravitons; as shown in figure 1. The two-point self-energy diagram is shown in figure 2, the

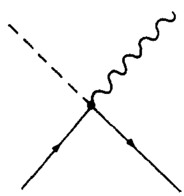

Figure 1. Electron–photon–graviton vertex in gravity-modified quantum electrodynamics. Full curve, electron; broken curve, gravitons; wavy curve, photon.

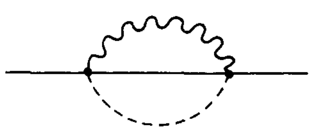

Figure 2. Gravity-modified self-energy part of order e^2.

corresponding matrix element is given by

$$\mathrm{i}\Sigma(x) = e^2 \left\langle 0 \left| TA_\mu(x)\frac{L^{\mu a}(x)}{\det L(x)}\gamma_a\psi(x)A_\nu(0)\frac{L^{\nu b}(0)}{\det L(0)}\bar{\psi}(0)\gamma_b \right| 0 \right\rangle$$
$$= e^2\langle 0|T\gamma_a A_\mu(x)\psi(x)\bar{\psi}(0)A_\nu(0)\gamma_b|0\rangle \mathscr{G}^{\mu a,\nu b}(x). \tag{4}$$

This amplitude differs from that of conventional quantum electrodynamics by the presence of function $\mathscr{G}^{\mu a,\nu b}(x)$. This two-point function $\mathscr{G}^{\mu a,\nu b}(x)$ represents the exchange of a set of infinitely many gravitons. It is called the superpropagator, the basic ingredient of gravity-modified quantum electrodynamics. It is involved in every diagram of order $(e^2)^n$ where $n = 1, 2, \ldots, \infty$. It is defined as

$$\mathscr{G}^{\mu a,\nu b}(x) = \left\langle 0 \left| T\frac{L^{\mu a}(x)}{\det L(x)}\frac{L^{\nu b}(0)}{\det L(0)} \right| 0 \right\rangle. \tag{5}$$

$L^{\mu a}(x)$ can be expressed in terms of $\phi^{\mu a}(x)$ by relation (2); for massless gravitons,

$$\langle 0|T\phi^{\mu a}(x)\phi^{\nu b}(0)|0\rangle = \tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab}+\eta^{\mu b}\eta^{\nu a}-\eta^{\mu a}\eta^{\nu b})D(x) \tag{6}$$

where $D(x)$ denotes the zero-mass causal propagator $(-4\pi^2x^2)^{-1}$. This shows that propagator $\mathscr{G}^{\mu a,\nu b}(x)$ will involve the object $(-1/x^2)^n$. This object possesses a singularity at $x = 0$, as n increases the singularity becomes worse and worse. When we go over to momentum-space amplitude the difficulty arises in defining the Fourier transform of this object.

To solve this problem Abdus-Salam and Strathdee (1970), Volkov (1968) and others make use of the Gelfand and Shilov (1964) technique for taking Fourier transforms of a generalized function $(-1/x^2)^n$ for appropriate regions of x and n. Then they analytically continue the results for other regions. Lehmann and Pohlmeyer (1971) and Taylor (1971) have shown rigorously that the resulting amplitudes possess the correct analytic structure associated with the requirement of unitarity.

In order to be able to use the Gelfand–Shilov formula they first perform a Sommerfeld–Watson transformation to convert the sum into an integral. Then the propagator is given by the contour integral,

$$\mathscr{G}^{\mu a,\nu b}(x) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}\,z<0} \mathrm{d}z\Gamma(-z)\cos\pi z\{\eta^{\mu a}\eta^{\nu b}\mathscr{D}^{(0)}(z)$$
$$+\tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab}+\eta^{\mu b}\eta^{\nu a}-\eta^{\mu a}\eta^{\nu b})\mathscr{D}^{(1)}(z)\}(\kappa^2D(x))^z \tag{7}$$

where the scalar amplitudes $\mathscr{D}^{(0)}(z)$ and $\mathscr{D}^{(1)}(z)$ are given respectively by

$$\mathscr{D}^{(0)}(z) = \mathscr{D}(z)-\tfrac{1}{36}z(19z+53)\mathscr{D}(z-1)+\tfrac{5}{36}z(z-1)(z+2)^2\mathscr{D}(z-2)$$
$$\mathscr{D}^{(1)}(z) = -\tfrac{1}{9}z(z+8)\mathscr{D}(z-1)+\tfrac{1}{18}z(z-1)(z+2)^2\mathscr{D}(z-2)$$

with $\mathscr{D}(z)$ given by

$$\mathscr{D}(z) = (\tfrac{1}{2})^{z+2}\Gamma(z+1)\Gamma(z+4)\{\cos\pi z\, B(\tfrac{3}{2}, 4+z)F(\tfrac{3}{2}, z+4; z+\tfrac{11}{2}; -1)$$
$$+B(-\tfrac{1}{2}, z+4)F(-\tfrac{1}{2}, z+4; z+\tfrac{7}{2}; -1)-4B(-\tfrac{1}{2}, 2)F(\tfrac{3}{2}, 2; \tfrac{3}{2}; -1)$$
$$+2(z+3)B(\tfrac{1}{2}, 2)F(\tfrac{1}{2}, 2; \tfrac{5}{2}; -1)\}. \tag{8}$$

These amplitudes are analytic in the neighbourhood of $z = 0$ at which point they take the following values:

$$\mathscr{D}^{(0)}(0) = 1 \quad \text{and} \quad \mathscr{D}^{(1)}(0) = 0$$

$$\left.\frac{d\mathscr{D}^{(0)}(z)}{dz}\right|_{z=0} = 7{\cdot}5 \quad \text{and} \quad \left.\frac{d\mathscr{D}^{(1)}(z)}{dz}\right|_{z=0} = -0{\cdot}5. \tag{9}$$

Next, a Fourier transformation is performed. According to the investigation of Gelfand and Shilov (1964) the Fourier transform of $(1/x^2)^z$ is a well defined classical mathematical object whenever $0 < \mathrm{Re}\, z < 2$ and is given by

$$D(p^2, z) = \frac{1}{(2\pi)^4 \mathrm{i}} \int d^4x\, e^{\mathrm{i}px} (D(x))^z = -\mathrm{i}\frac{(-p^2)^{z-2}\pi(4\pi)^{2-2z}}{\sin(\pi z)\Gamma(z)\Gamma(z-1)} \tag{10}$$

for $p^2 < 0$. The Fourier transform of $(1/x^2)^n$, with n lying outside this region is defined by the analytic continuation of this function in the variable z.

For amplitudes in euclidean space $x^2 = -(x_4^2 + \boldsymbol{x}^2)$; the function

$$(D(x))^z = (-1/4\pi^2 x^2)^z$$

is real and positive; and if the contour lies such that the power of $D(x)$ satisfies the condition $0 < \mathrm{Re}\, z < 2$, then the superpropagator in p space is defined as

$$\mathscr{G}^{\mu a, \nu b}(p^2) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\, z<0} dz \Gamma(-z) \cos \pi z \{\eta^{\mu a}\eta^{\nu b}\mathscr{D}^{(0)}(z)$$
$$+ \tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab} + \eta^{\mu b}\eta^{\nu a} - \eta^{\mu a}\eta^{\nu b})\mathscr{D}^{(1)}(z)\} D(p^2, z)(\kappa^2)^z \tag{11}$$

where $D(p^2, z)$ is given by relation (10). It can be rewritten as

$$D(p^2, z) = \mathrm{i}\frac{\Gamma(2-z)}{\Gamma(z)}(4\pi)^{2-2z}(-p^2)^{z-2} \qquad \text{for } p^2 < 0.$$

We always start perturbation theory computations in the Symanzik region of external momenta where the scalar products of all momenta are space-like. Since in general there are no production thresholds we can easily achieve this by a Wick rotation to euclidean space–time. After computations have been performed, we analytically continue to the physical subspace of momenta.

3. Self-mass of order e^4

For the conventional Dirac lagrangian

$$\mathscr{L}_{\mathrm{int}} = e\bar{\psi}\gamma_\mu \psi A_\mu$$

the Feynman diagrams, which contribute to the fourth order in the self-energy part, are shown in figure 3. These diagrams have been computed by Frank (1951) and Bialyncika-Birula (1965). By introducing gravitational interaction at each vertex and joining these vertices, pairwise, by superlines; we get the corresponding gravity-modified self-energy diagrams of order e^4 shown in figure 4.

Since our interest is to see how the regularization is going to proceed in higher order in e in gravity-modified quantum electrodynamics; we will compute the diagram (4c),

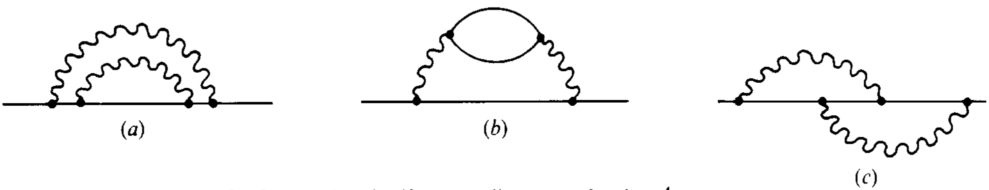

Figure 3. Conventional self-energy diagrams of order e^4.

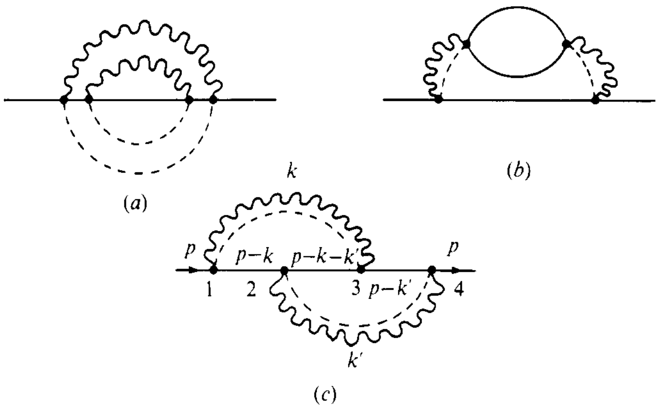

Figure 4. Gravity-modified self-energy diagrams of order e^4.

which seems to be most complicated, in detail. To be sure we have also checked the regularization for the other diagrams as well; these have all been individually regularized.

The matrix element corresponding to diagram (4*c*) is given by

$$\Sigma_4(x) = (ie)^4 \gamma_\mu S(x_1-x_2)\gamma_\nu S(x_2-x_3)\gamma_\lambda S(x_3-x_4)\gamma_\rho D_{ab}(x_1-x_3)D_{cd}(x_2-x_4) \times \mathscr{G}^{\mu a,\lambda b}(x_1-x_3)\mathscr{G}^{\nu c,\rho d}(x_2-x_4). \tag{12}$$

Substituting for $D_{ab}(x_i-x_j) = g_{ab}D(x_i-x_j)$ and $\mathscr{G}^{\mu a,\nu b}(x_i-x_j)$ given by relations (7) and (8) we obtain

$$\begin{aligned}\Sigma_4(x) = {} & (ie)^4 \gamma_\mu S(x_1-x_2)\gamma_\nu S(x_2-x_3)\gamma_\mu S(x_3-x_4)\gamma_\nu \\ & \times \int \frac{\mathrm{d}z_{13}}{2\pi \mathrm{i}}\Gamma(-z_{13})\cos \pi z_{13}(\mathscr{D}^{(0)}(z_{13})+2\mathscr{D}^{(1)}(z_{13}))(\kappa^2)^{z_{13}}(D(x_1-x_3))^{z_{13}+1} \\ & \times \int \frac{\mathrm{d}z_{24}}{2\pi \mathrm{i}}\Gamma(-z_{24})\cos \pi z_{24}(\mathscr{D}^{(0)}(z_{24})+2\mathscr{D}^{(1)}(z_{24}))(\kappa^2)^{z_{24}}(D(x_2-x_4))^{z_{24}+1}.\end{aligned} \tag{13}$$

Now we will go over to momentum space where calculations seem to be simple. For this purpose we take the Fourier transform of the above amplitude for euclidean space–time. Making use of the Gelfand–Shilov formula,

$$(D(x))^{z+1} = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, \mathrm{e}^{-\mathrm{i}kx}D(k^2, z), \tag{14}$$

where

$$D(k^2, z) = \mathrm{i}\frac{(-k^2)^{z-1}}{\Gamma(z+1)}(4\pi)^{-2z}\Gamma(1-z).$$

We can write the momentum-space amplitude for $k^2 < 0$ and $0 < \mathrm{Re}\, z < 2$

$$\Sigma_4(p) = \frac{e^4}{(2\pi)^8}\int \frac{dz_{13}}{2\pi i}F(z_{13})\int \frac{dz_{24}}{2\pi i}F(z_{24})\Sigma_4(p^2, z)$$

where

$$F(z_{ij}) = \Gamma(-z_{ij})\frac{\Gamma(1-z_{ij})}{\Gamma(1+z_{ij})}\cos \pi z_{ij}(\mathscr{D}^{(0)}(z_{ij})+2\mathscr{D}^{(1)}(z_{ij}))(\kappa^2/16\pi^2)^{z_{ij}}$$

and

$$\Sigma_4(p^2, z) = \int d^4k\, d^4k'(-k^2)^{z_{13}-1}(-k'^2)^{z_{24}-1} \times \frac{\gamma_\mu(\not p-\not k+m)\gamma_\nu(\not p-\not k-\not k'+m)\gamma_\mu(\not p-\not k'+m)\gamma_\nu}{\{(p-k)^2-m^2\}\{(p-k-k')^2-m^2\}\{(p-k')^2-m^2\}}. \tag{14}$$

Simplifying the numerator we can write

$$\Sigma_4(p) = mA(p^2)+\not pB(p^2), \tag{15}$$

where

$$\begin{aligned} A(p^2) = {} & \frac{e^4}{(2\pi)^8}\int \frac{dz_{13}}{2\pi i}F(z_{13})\int \frac{dz_{24}}{2\pi i}F(z_{24}) \\ & \times 4\{m^2 Y_0(p^2, z_{13}, z_{24})+Y_1(p^2, z_{13}, z_{24})+Y_2(p^2, z_{13}, z_{24}) \\ & + Y_3(p^2, z_{13}, z_{24})+Y_4(p^2, z_{13}, z_{24})+Y_5(p^2, z_{13}, z_{24})\} \\ B(p^2) = {} & \frac{e^4}{(2\pi)^8}\int \frac{dz_{13}}{2\pi i}F(z_{13})\int \frac{dz_{24}}{2\pi i}F(z_{24})\frac{1}{p^2}[2(m^2-p^2)\{Y_1(p^2, z_{13}, z_{24})+Y_2(p^2, z_{13}, z_{24}) \\ & + Y_3(p_2, z_{13}, z_{24})+Y_4(p^2, z_{13}, z_{24})\}-4p^2 Y_5(p^2, z_{13}, z_{24}) \\ & +4Y_6(p^2, z_{13}, z_{24})+4Y_7(p^2, z_{13}, z_{24})-4Y_8(p^2, z_{13}, z_{24}) \\ & -2Y_9(p^2, z_{13}, z_{24})-2Y_{10}(p^2, z_{13}, z_{24})], \end{aligned} \tag{16}$$

where

$$Y_0(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}-1}}{(p-k)^2-m^2}\frac{(-k'^2)^{z_{24}-1}}{(p-k')^2-m^2}\frac{1}{(p-k-k')^2-m^2}$$

$$Y_1(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}-1}}{(p-k)^2-m^2}\frac{(-k'^2)^{z_{24}}}{(p-k')^2-m^2}\frac{1}{(p-k-k')^2-m^2}$$

$$Y_2(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}}}{(p-k)^2-m^2}\frac{(-k'^2)^{z_{24}-1}}{(p-k')^2-m^2}\frac{1}{(p-k-k')^2-m^2}$$

$$Y_3(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}-1}}{(p-k)^2-m^2}\frac{(-k'^2)^{z_{24}-1}}{(p-k-k')^2-m^2}$$

$$Y_4(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}-1}}{(p-k')^2-m^2}\frac{(-k'^2)^{z_{24}-1}}{(p-k-k')^2-m^2}$$

$$Y_5(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}-1}}{(p-k)^2-m^2}\frac{(-k'^2)^{z_{24}-1}}{(p-k')^2-m^2}$$

$$Y_6(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}-1}}{(p-k)^2-m^2} \frac{(-k'^2)^{z_{24}-1}}{(p-k')^2-m^2} p\cdot k'$$

$$Y_7(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}-1}}{(p-k)^2-m^2} \frac{(-k'^2)^{z_{24}-1}}{(p-k')^2-m^2} p\cdot k$$

$$Y_8(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}-1}}{(p-k)^2-m^2} \frac{(-k'^2)^{z_{24}+1}}{(p-k')^2-m^2} \frac{1}{(p-k-k')^2-m^2}$$

$$Y_9(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}+1}}{(p-k)^2-m^2} \frac{(-k'^2)^{z_{24}-1}}{(p-k')^2-m^2} \frac{1}{(p-k-k')^2-m^2}$$

$$Y_{10}(p^2, z_{13}, z_{24}) = \int d^4k\, d^4k' \frac{(-k^2)^{z_{13}}}{(p-k)^2-m^2} \frac{(-k'^2)^{z_{24}}}{(p-k')^2-m^2} \frac{1}{(p-k-k')^2-m^2}. \tag{17}$$

Since k, k' and z_{13}, z_{24} are integration variables we can interchange k and k' and similarly z_{13} and z_{24}. Thus from (16) and (17) we can deduce the following relations: $Y_1(p^2) = Y_2(p^2)$, $Y_3(p^2) = Y_4(p^2)$, $Y_6(p^2) = Y_7(p^2)$ and $Y_8(p^2) = Y_9(p^2)$. Thus we need to evaluate only Y_0, Y_1, Y_3, Y_5, Y_6, Y_8 and Y_{10}. These amplitudes are evaluated in detail in the appendix of Parveen (1971), the results being as follows:

$$Y_0(p^2, z_{13}, z_{24}) = \pi^4 \frac{\Gamma(1+z_{24})}{\Gamma(1-z_{13})} \Gamma(1-z_{13}-z_{24}) \int_0^1 d\xi_1\, d\xi_2\, d\xi_3 (1-\xi_3)^{-z_{13}} \xi_1^{-z_{13}}$$
$$\times \{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-1-z_{24}}$$
$$\times \{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}^{z_{13}+z_{24}-1}$$
$$\times F\left(1+z_{24}, 1-z_{13}-z_{24};\right.$$
$$\left. 2; \frac{-\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2 p^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}}\right)$$

$$Y_1(p^2, z_{13}, z_{24}) = \pi^4 \frac{\Gamma(2+z_{24})}{\Gamma(1-z_{13})} \Gamma(-z_{13}-z_{24}) \int_0^1 d\xi_1\, d\xi_2\, d\xi_3\, \xi_1^{-z_{13}} (1-\xi_3)^{-z_{13}}$$
$$\times \{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-2-z_{24}}$$
$$\times \{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}^{z_{13}+z_{24}}$$
$$\times F\left(2+z_{24}, -z_{13}-z_{24};\right.$$
$$\left. 2; \frac{-\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2 p^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}}\right).$$

$$Y_3(p^2, z_{13}, z_{24}) = -\pi^4 \frac{\Gamma(1+z_{24})}{\Gamma(1-z_{13})} \Gamma(-z_{13}-z_{24}) \int_0^1 d\xi_1\, d\xi_2\, \xi_1^{-z_{13}}$$
$$\times \{\xi_2(1-\xi_2)+\xi_1(1-\xi_1-\xi_2)\}^{-1-z_{24}}(1-\xi_2)^{-1-z_{13}}$$
$$\times \{(m^2-p^2)(1-\xi_1)(1-\xi_2)+(1-\xi_1-\xi_2)^2p^2\}^{z_{13}+z_{24}}$$
$$\times F\left(1+z_{24}, -z_{13}-z_{24}; 2; \frac{-\{\xi_2(1-\xi_2)+\xi_1(1-\xi_1-\xi_2)\}p^2}{(m^2-p^2)(1-\xi_1)(1-\xi_2)+(1-\xi_1-\xi_2)^2p^2}\right)$$

$$Y_5(p^2, z_{13}, z_{24})$$
$$= -\pi^4\Gamma(-z_{13})(m^2)^{z_{13}}\Gamma(1+z_{13})F(-z_{13}, 1-z_{13}; 2; p^2/m^2)$$
$$\times\ \Gamma(-z_{24})(m^2)^{z_{24}}\Gamma(1+z_{24})F(-z_{24}, 1-z_{24}; 2; p^2/m^2)$$

$$Y_6(p^2, z_{13}, z_{24})$$
$$= -\pi^4 p^2 \tfrac{1}{2}\Gamma(-z_{13})\Gamma(1+z_{13})(m^2)^{z_{13}}F(-z_{13}, 1-z_{13}; 2; p^2/m^2)$$
$$\times\Gamma(-z_{24})\Gamma(2+z_{24})(m^2)^{z_{24}}F(-z_{24}, 1-z_{24}; 3; p^2/m^2)$$

$$Y_8(p^2, z_{13}, z_{24}) = \pi^4\frac{\Gamma(3+z_{24})}{\Gamma(1-z_{13})}\Gamma(-1-z_{13}-z_{24})\int_0^1 d\xi_1\, d\xi_2\, d\xi_3\, \xi_1^{-z_{13}}(1-\xi_3)^{-z_{13}}$$
$$\times\ \{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-3-z_{24}}$$
$$\times\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}^{z_{13}+z_{24}+1}$$
$$\times F\Bigg(3+z_{24}, -z_{13}-z_{24}-1;$$
$$2; \frac{-\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2p^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}}\Bigg)$$

$$Y_{10}(p^2, z_{13}, z_{24})$$
$$= \pi^4\frac{\Gamma(2+z_{24})}{\Gamma(-z_{13})}\Gamma(-1-z_{13}-z_{24})\int_0^1 d\xi_1\, d\xi_2\, d\xi_3(\xi_1)^{-z_{13}-1}(1-\xi_3)^{-z_{13}-1}$$
$$\times\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-2-z_{24}}$$
$$\times\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}^{z_{13}+z_{24}+1}$$
$$\times F(2+z_{24}, -1-z_{13}-z_{24}; 2; \text{same as in } Y_8). \tag{18}$$

The contour integration over z_{13} and z_{24} gives

$$Y_0(p^2) = \pi^4\int_0^1 d\xi_1\, d\xi_2\, d\xi_3\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^{-2}p^{-2}\ln\Bigg|1$$
$$+\frac{\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2p^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}}\Bigg|$$

$$Y_1(p^2) = -\pi^4\left(-\ln\frac{16\pi^2}{\kappa^2m^2}+\frac{15}{2}-3\psi(1)\right)\int_0^1\frac{d\eta}{\eta}$$
$$-\pi^4\int_0^1 d\xi_1\, d\xi_2\, d\xi_3\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-2}$$
$$\times\Bigg(\ln\frac{(1-p^2/m^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2/m^2}{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)}+\ln\Bigg|1$$
$$+\frac{\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2p^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}}\Bigg|$$

$$Y_3(p^2) = \pi^4\left(-\ln\frac{16\pi^2}{\kappa^2 m^2}+5{\cdot}5-3\psi(1)\right)(\tfrac{1}{3}\psi(4)-\tfrac{1}{3}\psi(1))$$

$$+\pi^4\int_0^1 d\xi_1\, d\xi_2(1-\xi_2)^{-1}\{\xi_2(1-\xi_2)+\xi_1(1-\xi_1-\xi_2)\}^{-1}$$

$$\times\left(\ln\frac{(1-p^2/m^2)(1-\xi_1)(1-\xi_2)+(1-\xi_1-\xi_2)^2p^2/m^2}{\xi_2(1-\xi_2)+\xi_1(1-\xi_1-\xi_2)}\right.$$

$$+m^2/p^2(1-\xi_1)(1-\xi_2)\{\xi_2(1-\xi_2)+\xi_1(1-\xi_1-\xi_2)\}^{-1}$$

$$\left.\times\ln\left|\frac{(1-\xi_1)(1-\xi_2)}{(1-p^2/m^2)(1-\xi_1)(1-\xi_2)+(1-\xi_1-\xi_2)^2p^2/m^2}\right|\right)$$

$$Y_5(p^2) = -\pi^4\left(-\ln\frac{16\pi^2}{\kappa^2 m^2}+5{\cdot}5-3\psi(1)-\frac{m^2-p^2}{p^2}\ln\left|\frac{m^2-p^2}{m^2}\right|\right)^2$$

$$Y_6(p^2) = -\pi^4 p^2\frac{1}{2}\left\{\left(\ln\frac{16\pi^2}{\kappa^2 m^2}\right)^2-(U_1+U_2)\ln\frac{16\pi^2}{\kappa^2 m^2}+U_1U_2\right\},$$

where

$$U_1 = 6{\cdot}5-3\psi(1)-\frac{m^2-p^2}{p^2}\ln\left|\frac{m^2-p^2}{m^2}\right|-1$$

$$U_2 = 7{\cdot}5-3\psi(1)-\frac{3}{2}+\frac{m^2}{p^2}-\frac{(m^2-p^2)^2}{p^4}\ln\left|\frac{m^2-p^2}{m^2}\right|$$

$$Y_8(p^2) = -\pi^4\left(\ln\frac{16\pi^2}{\kappa^2 m^2}\right)\left\{(m^2-p^2)\left(-\tfrac{1}{2}+\tfrac{5}{4}\ln 2+\tfrac{1}{4}\int_0^1\frac{d\eta}{\eta}\right)\right.$$

$$\left.+p^2\left(\tfrac{1}{12}-\tfrac{1}{12}\ln 2+\tfrac{7}{12}\int_0^1\frac{d\eta}{\eta}\right)\right\}$$

$$+\pi^4\int_0^1 d\xi_1\, d\xi_2\, d\xi_3\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-3}$$

$$\times\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}\left(U_3+U_4\right.$$

$$-\frac{3\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2p^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}}$$

$$\left.\times\ln\frac{16\pi^2}{\kappa^2 m^2}\right)$$

$$U_3 = 12{\cdot}5-9\psi(1)+2p^2\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3\}$$

$$\times\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-1}$$

$$\times\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}^{-1}\ln\left|1\right.$$

$$\left.+\frac{p^2\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2p^2\}}\right|$$

$$U_4 = 2\left(1+\tfrac{3}{2}\right.$$

$$\left.\times \frac{\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2 p^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}\{(m^2-p^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2 p^2\}}\right)$$

$$\times \ln\left|\frac{(1-p^2/m^2)(1-\xi_1)(1-\xi_3)+(1-\xi_1-\xi_3)^2 p^2/m^2}{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)}\right|$$

$$Y_{10}(p^2) = 0. \tag{19}$$

In the above expressions all the terms are finite with the exception of terms involving the η integral. This integral shows a divergence at $\eta = 0$; this is the usual infrared divergence and can be eliminated in the usual way (Jauch and Rohrlich 1955) for the real processes.

In general the object of physical interest is the fermion electromagnetic mass; this can be found from the expression for the self-energy part by substituting the mass-shell condition $\not p = m$ and $p^2 = m^2$ with

$$\lim_{p^2\to m^2} (m^2-p^2)\ln(m^2-p^2) = 0.$$

Thus from expressions (15), (16) and (19) we finally obtain a contribution of order e^4 to the electromagnetic mass.

$$\frac{1}{m}\delta^{(4)}(m) = \frac{1}{m}\Sigma_4(\not p = m) = \frac{4\pi^4 e^4}{(2\pi)^8}\left\{-\left(\ln\frac{16\pi^2}{\kappa^2 m^2}\right)^2 + \left(\tfrac{31}{12}\int_0^1 \frac{d\eta}{\eta} - \tfrac{1}{12}\ln 2 + 6\psi(1) - \tfrac{2}{3}\psi(4)\right.\right.$$

$$\left.\left. + \tfrac{2}{3}\psi(1) - 12{\cdot}5 + \tfrac{1}{12}\right)\ln\frac{16\pi^2}{\kappa^2 m^2}\right\} + \left[(3\psi(1)-6{\cdot}5)\int_0^1 \frac{d\eta}{\eta}\right.$$

$$+ \tfrac{2}{3}(5{\cdot}5 - 3\psi(1))(\psi(4)-\psi(1)) + 2(12{\cdot}5 - 9\psi(1))\left(-\tfrac{1}{12}\log 2\right.$$

$$\left.\left. + \tfrac{7}{12}\int_0^1 \frac{d\eta}{\eta} + \tfrac{1}{12}\right) + (5{\cdot}5 - 3\psi(1))(7 - 3\psi(1)) + I^{(c)}\right],$$

where

$$\psi(1+n) = 1/n + \psi(n), \qquad \psi(1) = -0{\cdot}577$$

$$I^{(c)} = I^{(0)} + I^{(1)} + I^{(3)} + I^{(8)}.$$

$$I^{(0)} = \int_0^1 d\xi_1\, d\xi_2\, d\xi_3 \{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^{-2}$$

$$\times \ln\left|1 + \frac{\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}\{(1-\xi_1-\xi_3)^2\}}\right|$$

$$I^{(1)} = -2\int_0^1 d\xi_1\, d\xi_2\, d\xi_3 \{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-2}$$

$$\times \left(\ln\left|\frac{(1-\xi_1-\xi_3)^2}{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)}\right|\right.$$

$$\left. + \ln\left|1 + \frac{\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}(1-\xi_1-\xi_3)^2}\right|\right)$$

$$I^{(3)} = 2\int_0^1 d\xi_1\, d\xi_2 (1-\xi_2)^{-1}\{\xi_2(1-\xi_2)+\xi_1(1-\xi_1-\xi_2)\}^{-1}$$

$$\times\left(\ln\left|\frac{(1-\xi_1-\xi_2)^2}{\xi_2(1-\xi_2)+\xi_1(1-\xi_1-\xi_2)}\right| + (1-\xi_1)(1-\xi_2)\right.$$

$$\left.\times\{\xi_2(1-\xi_2)+\xi_1(1-\xi_1-\xi_2)\}^{-1}\ln\left|\frac{(1-\xi_1)(1-\xi_2)}{(1-\xi_1-\xi_2)^2}\right|\right)$$

and

$$I^{(8)} = 2\int_0^1 d\xi_1\, d\xi_2\, d\xi_3 \{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-3}$$

$$\times\Bigg\{\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}2\{\xi_3(1-\xi_3)$$

$$+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}^{-1}$$

$$\times\ln\left|1+\frac{\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2}{\{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)\}(1-\xi_1-\xi_3)^2}\right|$$

$$-3\frac{\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}}{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)}\ln\frac{16\pi^2}{\kappa^2 m^2}$$

$$+2\left((1-\xi_1-\xi_3)^2+\tfrac{3}{2}\frac{\{\xi_3(1-\xi_3)+\xi_1(1-\xi_1-\xi_2-\xi_3)\}^2}{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)}\right)$$

$$\times\ln\left|\frac{(1-\xi_1-\xi_3)^2}{\xi_3(1-\xi_3)+(\xi_1+\xi_2)(1-\xi_1-\xi_2-\xi_3)}\right|\Bigg\}.$$

In addition to these terms there will be other contributions which are of order $(\kappa^2)^n$, going to zero with κ^2.

4. Conclusion

The fourth-order contribution to the self-mass of the electron is finite in gravity-modified quantum electrodynamics. From the point of view of the theory, it is very encouraging that for higher orders as well gravity acts as an inbuilt regulator. Thus one can conclude that a matrix element, computed to any higher order in e (to all orders in κ), is finite. The consistency of the theory is obvious from the fact that if we take the zero-gravity limit in the above result, the old logarithmic infinities (Frank 1951) of the type $(\ln\infty)^2$ and $\ln\infty$, reappear.

Acknowledgments

The author wishes to thank Professor Abdus-Salam, Drs J Strathdee and C J Isham for their constant help, guidance and encouragement. She wishes to express her appreciation of the stimulating atmosphere provided by both staff and scientists present at the International Centre for Theoretical Physics, Trieste, Italy, where this work was done. The financial assistance of the Ministry of Education, Pakistan Government, is gratefully acknowledged.

References

Abdus-Salam, Isham C J and Strathdee J 1971 *Phys. Rev.* D **3** 1805
Abdus-Salam and Strathdee J 1970 *Phys. Rev.* D **1** 3296
Bialyncika-Birula Z 1965 *Bull. Acad. pol. Sci. Math.* **13** 369
Efimov G V 1963 *Sov. Phys.-JETP* **17** 1417
Fradkin E S 1963 *Nucl. Phys.* **49** 624
Frank R M 1951 *Phys. Rev.* **83** 1189
Gelfand I M and Shilov G E 1964 *Generalized Functions* vol 1 (New York: Academic Press)
Jauch J M and Rohrlich F 1955 *The Theory of Photons and Electrons* (Reading, Mass: Addison Wesley)
Lehmann H and Pohlmeyer K 1971 *Commun. math. Phys.* **20** 101
Parveen S 1971 *PhD Thesis* Imperial College, London University pp 226–47
Taylor J G 1971 *University of Southampton preprint*
Volkov M K 1968 *Ann. Phys., NY* **49** 269